Negative Resistance Oscillators Revisited

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-ABSTRACT-

Negative resistance oscillator theory is typically introduced to undergraduate electrical engineers using a quasi-linear analysis based on transfer functions for what is , by essence, a non linear phenomenon. A more rigorous treatment based on differential equations is presented in Chua-Desoer and Kuh, and while it sheds a significant amount of light on the nonlinear dynamics involved , it fails to give an elementary treatment of what is after all the essential difficulty in low distortion sinusoidal oscillator theory: amplitude and distortion evaluation. In this work, under an assumption of low distortion, we compute an asymptotic approximation of the amplitude of oscillation based on an elementary analysis of the evolution of the total energy stored in energy storing circuit elements over a cycle. A transcendental equation with normalized variables is obtained which shows that under low distortion conditions, the amplitude of inductance tension can vary from a minimum of V_{SAT} (minimal distortion) to a maximum of $(4/\pi)V_{SAT}$ (maximal distortion), where V_{SAT} is the operational amplifier saturation voltage for the amplifier used to synthesize the negative resistance. Estimates of third harmonic distortion levels are also given .

1. INTRODUCTION

It has long been understood that building electric oscillations requires an active circuit (providing a source of energy when needed as well as a potentially non linear behavior) with an unstable equilibrium when voltages and currents are zero, together with a non linear mechanism which acts as a stabilizing force when the amplitude of the signals of interest starts growing sufficiently. For sinusoidal oscillators, it is essential to have a section of the circuit which acts as a frequency selective component. Note that under successful low distortion sinusoidal conditions, linear sinusoidal analysis can be used to identify possible frequencies of oscillations. This is not the case for amplitude computation. Note also that non linear mechanisms are essential given that linear time-invariant circuits cannot be at once stable and unstable according to the operating point of the circuit.

Our aim in this paper is to develop an elementary treatment of amplitude analysis of LC negative resistance oscillators, where the piecewise linear negative resistance is synthesized using an operational amplifier and linear resistors. The analysis is based on low distortion assumptions (pure sinusoidal signals at the natural frequency of the LC circuit) and a requirement that total energy stored in the circuit be a periodic function of time at the very period defined by the LC circuit. Approximate third harmonic distortion levels are estimated *a posteriori*, and the impact of each resistor forming the negative resistance is assessed.

2. AMPLITUDE CALCULATIONS

We consider the following candidate circuit for sinusoidal oscillations :

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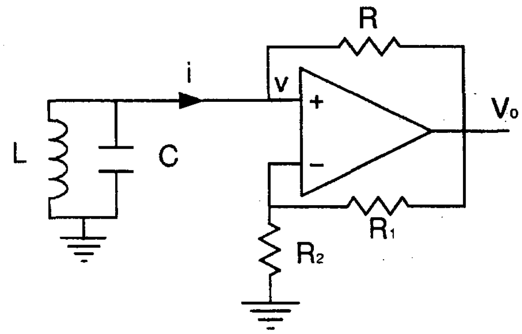

Fig. 1

It is easy to show that the i-v characteristic is as follows :

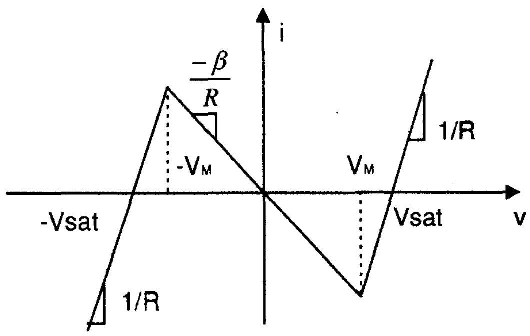

Fig. 2

where $\beta = R_1 / R_2$, $V_M = Vsat / (1 + \beta)$ and Vsat is the saturation voltage of the amplifier.
Under low distortion conditions, the LC circuit dictates the frequency of oscillations $\omega_0 = (LC)^{-1/2}$. Thus, if we assume that $v\ \upsilon\ A \sin\omega_0 t$, i.e. we fully neglect any distortion, the question that arises is what should be the amplitude of A ?

Equivalently, we require that :

$$\int_0^T \frac{dE}{d\tau} d\tau = 0 \qquad (1)$$

We attempt to answer this question by arguing that the total energy stored in the circuit must be periodic with period $T = (2\pi / \omega_0)$.

where E (τ) is the instantaneous energy stored in the circuit. But $E(\tau) = (1/2)\, Cv^2(\tau) + (1/2) L i_L^2(\tau)$, whereas $dE/d\tau = -vf(v)$ (2), where f(v) is the piecewise linear characteristic in Fig.2.
Using (1), (2) and the sinusoidal assumption, it is possible to obtain the following results :

- An analytic expression of the amplitude if f(v) is approximated by either a third order polynomial $f(v) = -a_1 v + a_3 v^3$, a_1, a_3 positive (this is in direct connection with the Van Der Pol equation), or a fifth order polynomial $f(v) = -a_1 v + a_3 v^3 + a_5 v^5$ a_1, a_3, a_5 positive. In the case of the Van Der Pol like equation, the result is $A = 2\,(a_1/a_3)^{1/2}$.

- A transcendental equation the solution of which is the amplitude. The equation has the form :

$$\frac{\pi}{2} - \theta(1+\beta) = (1+\beta)\frac{1}{2}\sin(2\theta) \qquad (3)$$

where $\theta = \sin^{-1}(V_M / A)$.

Equation (3) shows that the amplitude A evolves monotonically from A = Vsat when β is near zero to A= ($4/\pi$) Vsat when β goes to infinity (we recognize the latter as the first harmonic of a square wave with amplitude Vsat). Note that under our low distortion assumption, the amplitude of the oscillation is influenced only by the β factor and is thus *independent of resistor R*. Numerical results are shown below.

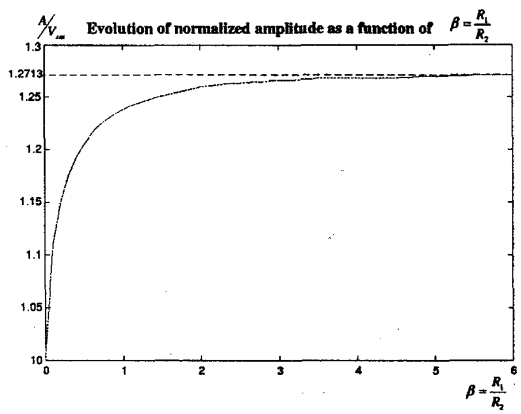

Fig. 3

3. DISTORTION ESTIMATES.

Based on the above results , we compute *a posteriori* the waveform v_0 and subsequently the approximate magnitude of the third order harmonic distortion in the signal through linear filtering analysis (this is the point where one realizes that the larger resistor R ,the less C will show up in signal v. Indeed, R affects directly the quality factor of the LCR resonant circuit). If the resulting distortion is too high, then the assumptions used throughout the analysis are invalid, and further iterations with different assumptions on the behavior of v may be needed.
The results of our calculations are as follows :

$$D_3^0 = 2\frac{V_{SAT}}{\pi}\left[1-\frac{\cos\theta}{3}\left(1+2\sin^2\theta\right)\right] \quad ,$$

$$D_3 = D_3^0\frac{3}{8}\frac{Q^{-1}}{\sqrt{1+\left(\frac{3}{8}\right)^2 Q^{-2}}}.$$

where D_3^0 and D_3 are the third harmonic distortion in V_0 and V respectively while Q is the quality factor of the R, L, C circuit at the frequency of oscillation.

4. CONCLUSION

We have presented an elementary treatment of the amplitude analysis problem in operational amplifier based negative resistance oscillators. The most important result is a new transcendental equation characterizing the dependence of amplitude on the size R_1/R_2 of negative feedback in the circuit under assumptions of low distortion. The role of the resistor R in the positive feedback loop as a distortion control factor has also been elucidated, and distortion estimates have been given. The computational techniques developed here appear to be promising for the analysis of both Wien and phase shift oscillators.

5. REFERENCES.

[1] L.O.Chua, C.A.Desoer,E.S.Kuh , *Linear and Nonlinear Circuits* ,Mc Graw-Hill,1987

APPENDIX – DERIVATIONS

A1- VAN DER POL OSCILLATOR

Application of Equation(2) yields:

$$\int_0^T \frac{dE}{d\tau}d\tau = \int_0^T (a_1A^2\sin^2\omega_0\tau \_ a_4A^4\sin^4\omega_0\tau)d\tau$$

$$= \int_0^T (a_1A^2\left(\frac{1-\cos 2\omega_0\tau}{2}\right) - a_3A^4\left(\frac{1-\cos 2\omega_0\tau}{2}\right)^4)d\tau$$

$$= 0 \qquad (A.1.1)$$

(A.1.1) yields:

$$A = 2\,(a_1/a_3)^{1/2} \qquad (A..1.2)$$

A2- PIECEWISE-LINEAR RESISTANCE

Application of Equation(2) , after the change of variable $u = A\sin\omega_0\tau$, yields:

$$\int_0^T \frac{dE}{d\tau}d\tau = -2\int_0^A \frac{uf(u)}{\omega_0\sqrt{(A^2-u^2)}}du = 0 \qquad (A.2.1)$$

If one now breaks the integral in (A.2.1) into the analytic segments of f(v), say $f^{(1)}(v)$ and $f^{(2)}(v)$, the result is as follows:

$$\int_0^A \frac{uf(u)}{\sqrt{(A^2-u^2)}}du = \int_0^{V_M} \frac{uf^{(1)}(u)}{\sqrt{(A^2-u^2)}}du$$

$$+ \int_{V_M}^A \frac{uf^{(2)}(u)}{\sqrt{(A^2-u^2)}}du$$

$$= -\sqrt{(A^2-u^2)}f^{(1)}(u)\Big|_0^{V_M}$$

$$+ \int_0^{V_M} \sqrt{(A^2-u^2)}f^{(1)'}(u)du$$

$$- \sqrt{(A^2-u^2)}f^{(2)}(u)\Big|_{V_M}^{A}$$

$$+ \int_{V_M}^A \frac{uf^{(2)'}(u)}{\sqrt{(A^2-u^2)}}du$$

$$= 0 \qquad (A.2.2)$$

(A.2.2) yields:

$$
\begin{aligned}
&-\sqrt{\left(A^2-V_M{}^2\right)}f^{(1)}(V_M)+Af^{(1)}(0)\\
&+\sqrt{\left(A^2-V_M{}^2\right)}f^{(2)}(V_M)\\
&-G_1\left(\frac{u}{2}\sqrt{\left(A^2-u^2\right)}+\frac{A^2}{2}\sin^{-1}(\frac{u}{A})\right)\Bigg|_0^{V_M}\\
&+G\left(\frac{u}{2}\sqrt{\left(A^2-u^2\right)}+\frac{A^2}{2}\sin^{-1}(\frac{u}{A})\right)\Bigg|_{V_M}^{A}\\
&=0 \qquad\qquad\qquad (A.2.3)
\end{aligned}
$$

In (A.2.3), $G_1=\beta/R$, and $G = 1/R$.
Note that $f^{(1)}(V_M) = f^{(1)}(0) = f^{(2)}(V_M) = 0$. This yields:

$$
\begin{aligned}
&-(1+\beta)\left[\frac{V_M}{2}\sqrt{A^2-V_M^2}+\frac{A^2}{2}\sin^{-1}\left(\frac{V_M}{A}\right)\right]\\
&+\frac{A^2\pi}{4}=0 \qquad\qquad\qquad (A.2.4)
\end{aligned}
$$

Equation (A.2.4) is essentially the same as equation (3). This completes our derivation.