Cramer's rule

In linear algebra, **Cramer's rule** is a theorem, which gives an expression for the solution of a system of linear equations with as many equations as unknowns, valid in those cases where there is a unique solution. The solution is expressed in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the vector of right hand sides of the equations. It is named after Gabriel Cramer (1704–1752), who published the rule in his 1750 *Introduction à l'analyse des lignes courbes algébriques*, although Colin Maclaurin also published the method in his 1748 Treatise of Algebra (and probably knew of the method as early as 1729).[1]

Explicit formulas for small systems

Consider the linear system $\begin{cases} ax+by &= {\color{red}e} \\ cx+dy &= {\color{red}f} \end{cases}$ which in matrix format is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} {\color{red}e} \\ {\color{red}f} \end{bmatrix}$. Then, x and y can be found with Cramer's rule as

$$x = \begin{vmatrix} {\color{red}e} & b \\ {\color{red}f} & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{{\color{red}e}d - b{\color{red}f}}{ad - bc} \text{ and}$$

$$y = \begin{vmatrix} a & {\color{red}e} \\ c & {\color{red}f} \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{a{\color{red}f} - {\color{red}e}c}{ad - bc}.$$

The rules for 3×3 are similar. Given $\begin{cases} ax+by+cz &= {\color{red}j} \\ dx+ey+fz &= {\color{red}k} \\ gx+hy+iz &= {\color{red}l} \end{cases}$ which in matrix format is

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} {\color{red}j} \\ {\color{red}k} \\ {\color{red}l} \end{bmatrix}$$ the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} {\color{red}j} & b & c \\ {\color{red}k} & e & f \\ {\color{red}l} & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & {\color{red}j} & c \\ d & {\color{red}k} & f \\ g & {\color{red}l} & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a & b & {\color{red}j} \\ d & e & {\color{red}k} \\ g & h & {\color{red}l} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

General case

Consider a system of linear equations represented in matrix multiplication form as follows:

$$Ax = b \qquad (1)$$

where the square matrix A is invertible and the vector $x = (x_1, \ldots, x_n)^\top$ is the column vector of the variables. Then the theorem states that:

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

where A_i is the matrix formed by replacing the ith column of A by the column vector b .

The rule holds for systems of equations with coefficients and unknowns in any field, not just in the real numbers. This formula is, however, of limited practical value for larger matrices, as there are other more efficient ways of solving systems of linear equations, such as by Gauss elimination or, even better, LU decomposition.

Finding inverse matrix

Let *A* be an *n*×*n* matrix. Then

$$\mathrm{Adj}(A)A = \det(A)I$$

where Adj(*A*) denotes the adjugate matrix of *A*, det(*A*) is the determinant, and I is the identity matrix. If det(*A*) is invertible in *R*, then the inverse matrix of *A* is

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A).$$

If *R* is a field (such as the field of real numbers), then this gives a formula for the inverse of *A*, provided det(*A*) ≠ 0.

Applications

Differential geometry

Cramer's rule is also extremely useful for solving problems in differential geometry. Consider the two equations $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$. When *u* and *v* are independent variables, we can define $x = X(u, v)$ and $y = Y(u, v)$.

Finding an equation for $\dfrac{\partial x}{\partial u}$ is a trivial application of Cramer's rule.

First, calculate the first derivatives of *F*, *G*, *x*, and *y*:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv = 0 \\ dG &= \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial u}du + \frac{\partial G}{\partial v}dv = 0 \\ dx &= \frac{\partial X}{\partial u}du + \frac{\partial X}{\partial v}dv \\ dy &= \frac{\partial Y}{\partial u}du + \frac{\partial Y}{\partial v}dv. \end{aligned}$$

Substituting *dx*, *dy* into *dF* and *dG*, we have:

Since *u*, *v* are both independent, the coefficients of *du*, *dv* must be zero. So we can write out equations for the coefficients:

$$\begin{aligned} \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} &= -\frac{\partial F}{\partial u} \\ \frac{\partial G}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial u} &= -\frac{\partial G}{\partial u} \\ \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} &= -\frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial v} &= -\frac{\partial G}{\partial v}. \end{aligned}$$

Now, by Cramer's rule, we see that:

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial u} & \frac{\partial F}{\partial y} \\ -\frac{\partial G}{\partial u} & \frac{\partial G}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}}.$$

This is now a formula in terms of two Jacobians:

$$\frac{\partial x}{\partial u} = -\frac{\left(\frac{\partial(F,G)}{\partial(u,y)}\right)}{\left(\frac{\partial(F,G)}{\partial(x,y)}\right)}.$$

Similar formulae can be derived for $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

Algebra

Cramer's rule can be used to prove the Cayley–Hamilton theorem of linear algebra, as well as Nakayama's lemma, which is fundamental in commutative ring theory.

Integer programming

Cramer's rule can be used to prove that an integer programming problem whose constraint matrix is totally unimodular and whose right-hand side has integer basic solutions. This makes the integer program substantially easier to solve.

Ordinary differential equations

Cramer's rule is used to derive the general solution to an inhomogeneous linear differential equation by the method of variation of parameters.

Proof

Cramer's rule can be proven using two properties of determinants only. The first property is that adding one column to another does not change the value of the determinant, and the second property is that multiplying every element of one column by a factor will multiply the value of the determinant by the same factor.

Given n linear equations with n variables $x_1, x_2, \ldots, x_n$.

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{matrix}$$

Cramer's rule gives, for the value of x_1, the expression:

$$\frac{\det\begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix}}{\det\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}}$$

which can be verified using the aforementioned properties of determinants. In fact, from the equations of the system, this quotient is equal to

$$\frac{\det\begin{vmatrix}(a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n) & a_{12} & \dots & a_{1n}\\ (a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n) & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ (a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n) & a_{n2} & \dots & a_{nn}\end{vmatrix}}{\det\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}}$$

By subtracting from the first column the second multiplied by x_2, the third column multiplied by x_3, and so on until the last column multiplied by x_n , it is found to be equal to

$$\frac{\det\begin{vmatrix}a_{11}x_1 & a_{12} & \dots & a_{1n}\\ a_{21}x_1 & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}x_1 & a_{n2} & \dots & a_{nn}\end{vmatrix}}{\det\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}},$$

and according to the remaining property of determinants this is equal to

$$\frac{x_1\det\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}}{\det\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}}=x_1.$$

In the same way, if the columns of **b**'s is replacing the **k**-th column of the matrix of the system of equations the result will be equal to x_k . As a result we get that

These expressions for x_m can be put into matrix notation as follows. First do a Laplace expansion (aka cofactor expansion) on the determinants which are in the numerators using the columns which contain $b_1,\ b_2,\ \cdots,\ b_n$. Thus Cramer's rule becomes;

$$x_m=\frac{c_{1m}b_1+c_{2m}b_2+\ldots+c_{nm}b_n}{det\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}}$$

Where c_{rc} are the cofactors of the coefficient matrix [A].

$and\ c_{rc}\ =-1^{r+c}M_{rc},\ where\ M_{rc}$is the determinant of the matrix formed by deleting row r and column c from [A]. Therefore Cramer's rule solutions for x_m has the matrix form

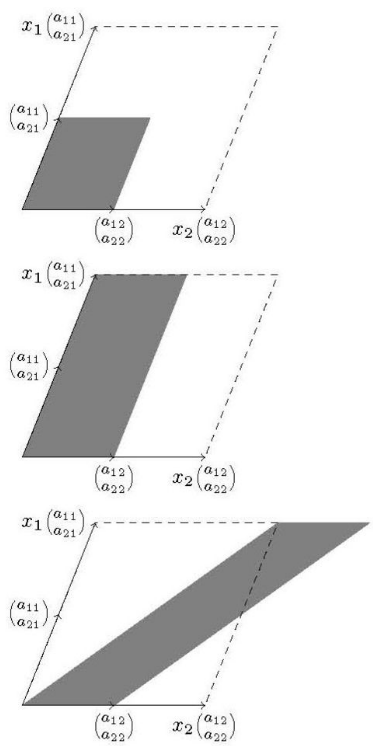
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}{det \begin{vmatrix} a_{11} & a_{12} & \dots & a_1n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}$$

$[C]^T$ is called the adjugate matrix of $[A]$, written as adj[A].

Geometric interpretation

Cramer's rule has a geometric interpretation that can be considered also a proof or simply giving insight about its geometric nature. These geometric arguments work in general and not only in the case of two equations with two unknowns presented here.

Given the system of equations



Geometric interpretation of Cramer's rule. The areas of the second and third shaded parallelograms are the same and the second is x_1 times the first. From this equality Cramer's rule follows.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

it can be considered as a equation between vectors

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The area of the parallelogram determined by $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$ is given by the determinant of the system of equations

$$\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

In general, when there are more variables and equations, the determinant of n vectors of length n will give the *volume* of the *parallelepiped* determined by those vectors in the n -th dimensional Euclidean space.

Therefore the area of the parallelogram determined by $x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$ has to be x_1 times the area of the first one since one of the sides has been multiplied by this factor. Now, this last parallelogram, by Cavalieri's principle, has the same area as the parallelogram determined by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$ and $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$.

Equating the areas of this last and the second parallelogram gives the equation

$$\det \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \det \begin{vmatrix} a_{11}x_1 & a_{12} \\ a_{21}x_1 & a_{22} \end{vmatrix} = x_1 \det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

from where Cramer's rule follows.

Incompatible and indeterminate cases

A system of equations is said to be incompatible when there are no solutions and it is called indeterminate when there is more than one solution. For linear equations, an indeterminate system will have infinitely many solutions (if it is over an infinite field), since the solutions can be expressed in terms of one or more parameters that can take arbitrary values.

Cramer's rule applies to the case where the coefficient determinant is nonzero. In the contrary case the system is either incompatible or indeterminate, based on the values of the determinants only for 2x2 systems.

For 3x3 or higher systems, the only thing one can say when the coefficient determinant equals zero is: if any of the "numerator" determinants are nonzero, then the system must be incompatible. However, the converse is false: having all determinants zero does not imply that the system is indeterminate. A simple example where all determinants vanish but the system is still incompatible is the 3x3 system x+y+z=1, x+y+z=2, x+y+z=3.

See also

- Matrix

Notes

[1] Carl B. Boyer, *A History of Mathematics*, 2nd edition (Wiley, 1968), p. 431.

External links

- Proof of Cramer's Rule (http://planetmath.org/encyclopedia/ProofOfCramersRule.html)

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